Rook-drawing for planar graphs

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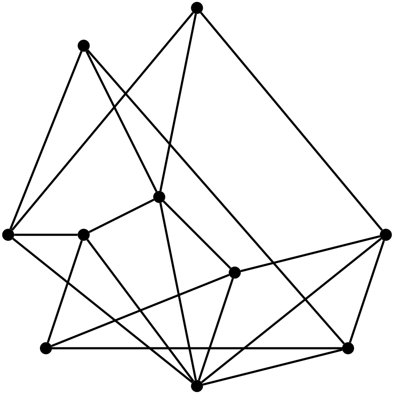
From joint work with David Auber, Nicolas Bonichon and Paul Dorbec

LaBRI, Bordeaux

January 16^{th}, 2015

Drawing graphs

How can we draw graphs without too much effort when the graph changes ?



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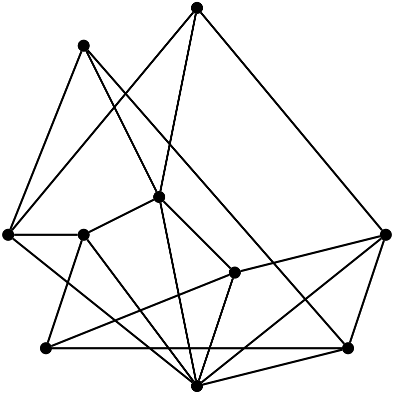

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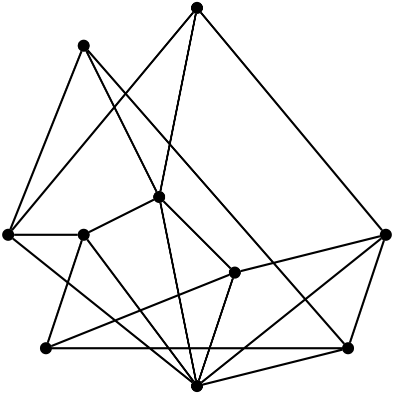

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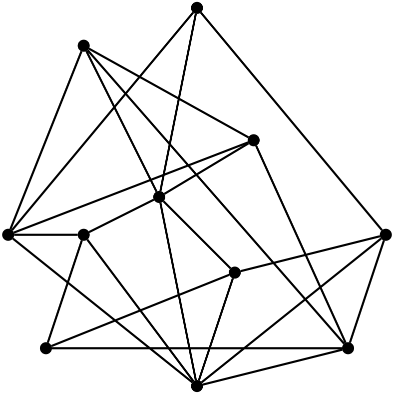

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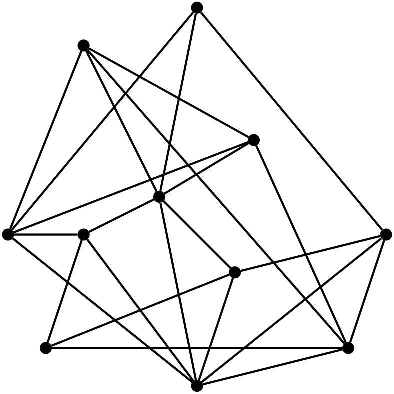

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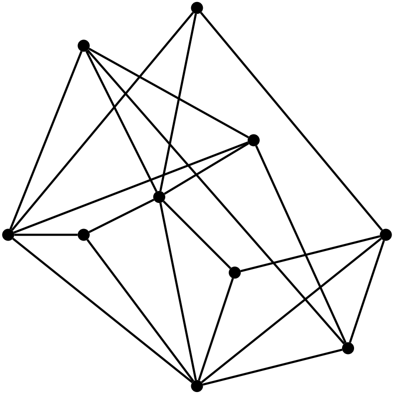

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- addition of nodes
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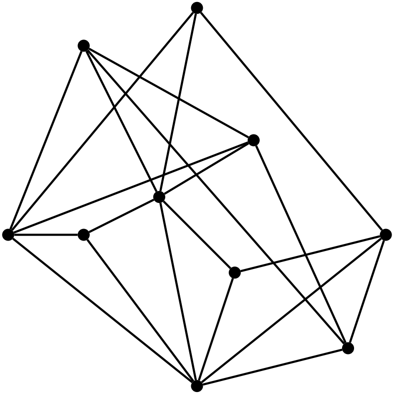
- node "expansion"



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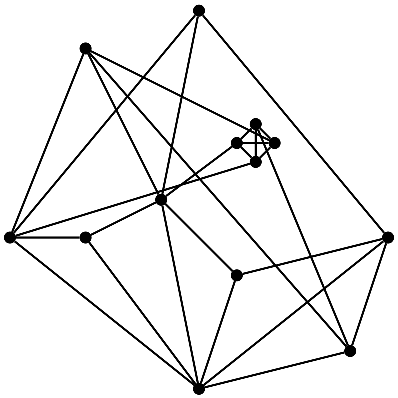

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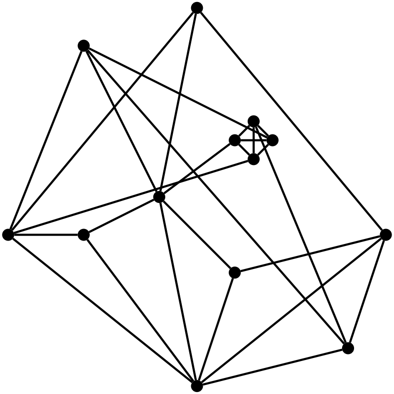

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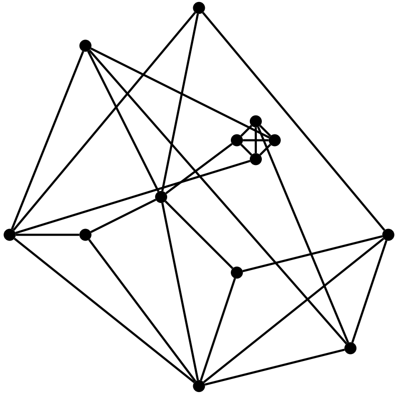

Drawing graphs

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- addition of nodes
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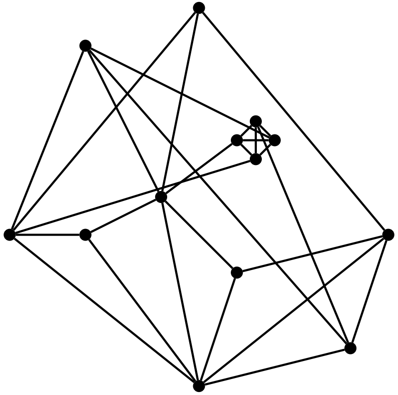
- Looking for linear algorithms, easy to implement
- Preservation of the global "image" of the graph

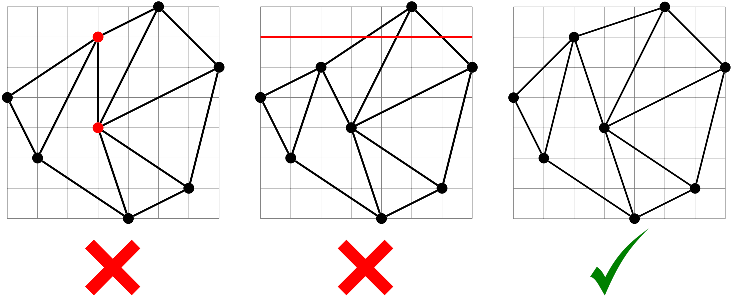


Rook-drawing

Definition

A rook-drawing of a graph of n vertices :

- Straight-lines
- Regular grid $n \times n$
- One vertex per line and column exactly



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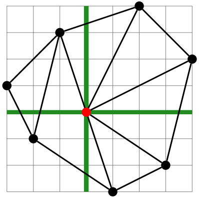

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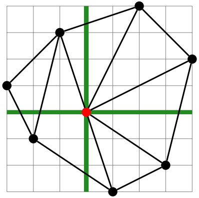

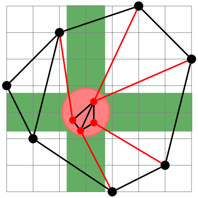

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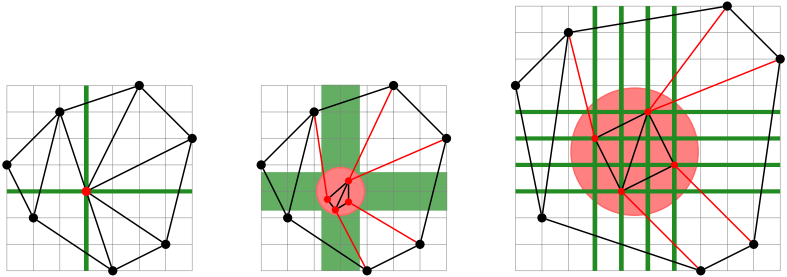

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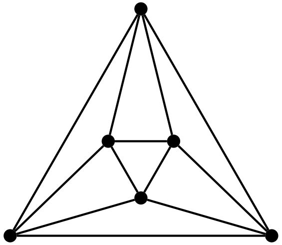

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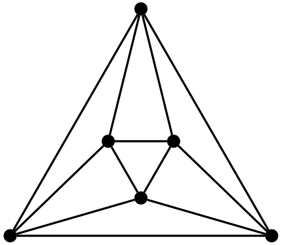

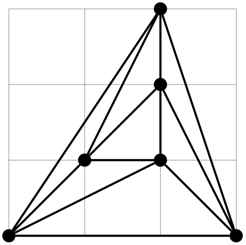

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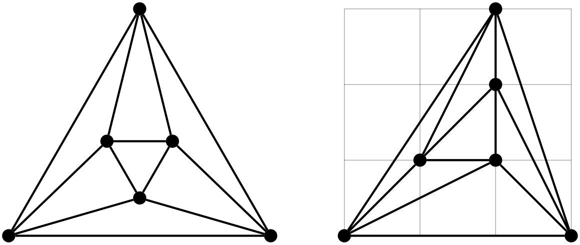
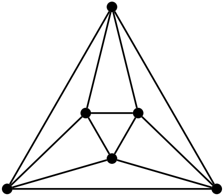


A counter-example



Three exterior nodes a, b and c. Inner nodes : inside the area (ab), (bc) and (ca).

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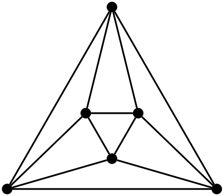

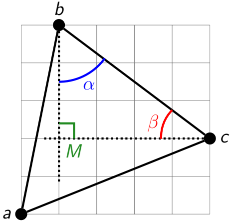

Three exterior nodes *a*, *b* and *c*. Inner nodes : inside the area (*ab*), (*bc*) and (*ca*).

$\alpha \geq 45°$, $\beta \geq 45°$. *Mab* right-angled

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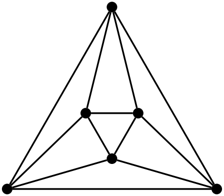

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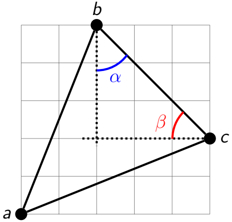

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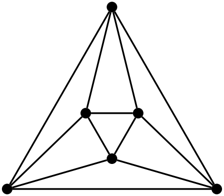

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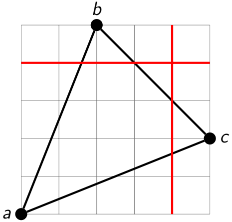

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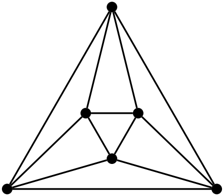

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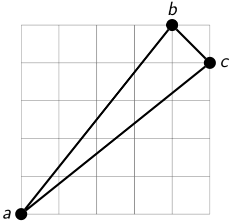

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Fill these line and column with c and b !
$x(b) = y(c) = n - 1$.

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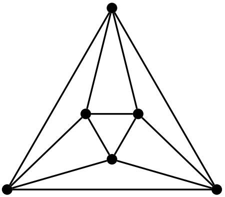

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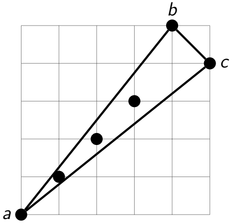

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Fill these line and column with c and b!
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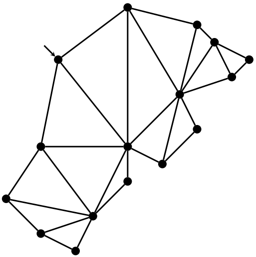
Inner nodes : along a diagonal.

Rook-drawing for outerplanar graphs

A graph is outerplanar if it has a planar drawing such that all its vertices are on the outer face.



Result

Every outerplanar graph has a rook-drawing which can be computed in linear time.

Rook-drawing for outerplanar graphs

[Bonichon, Gavoille, Hanusse, 2005]

- edges of G outerplanar map $\rightarrow$ T_r, T_b
- edges of T_r : between u and its first non-descendant found after u in a clockwise pre-ordering of T_b.

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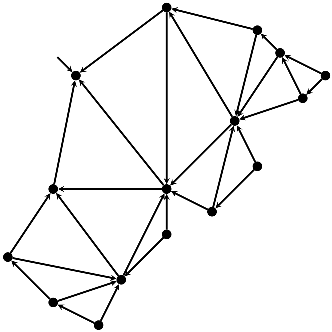

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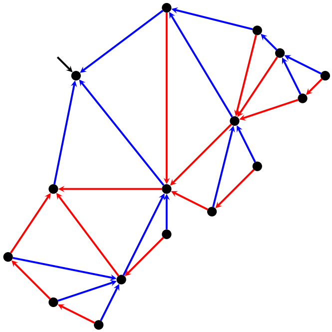

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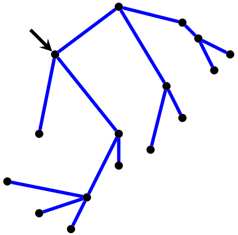

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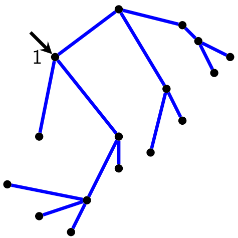

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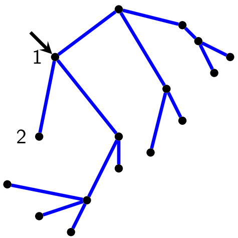

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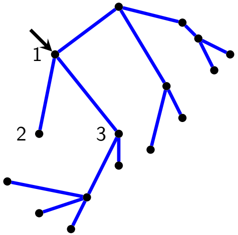

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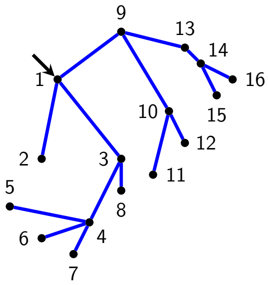

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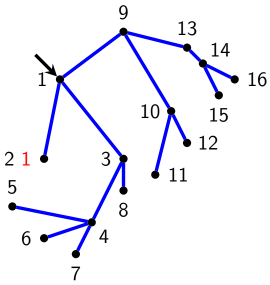

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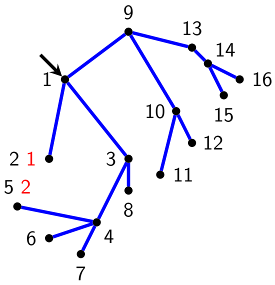

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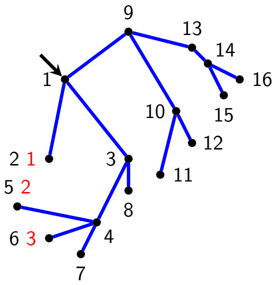

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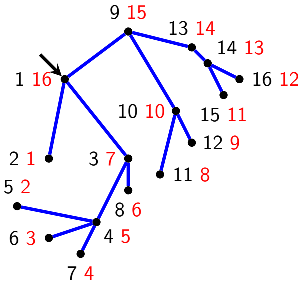

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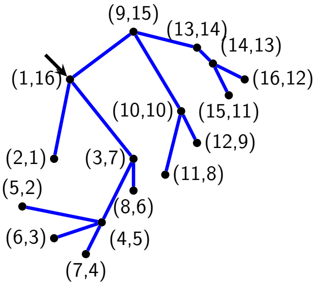

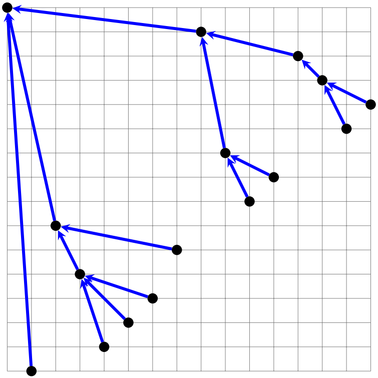

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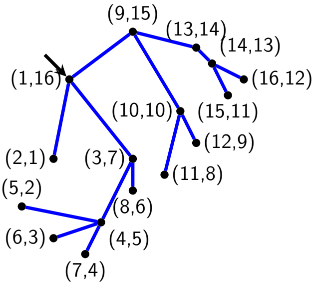

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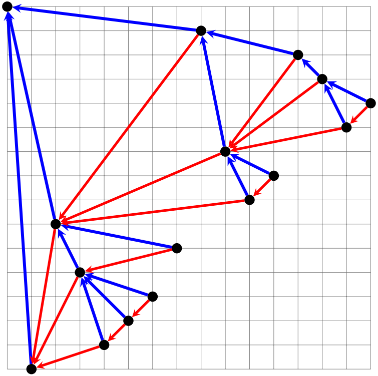

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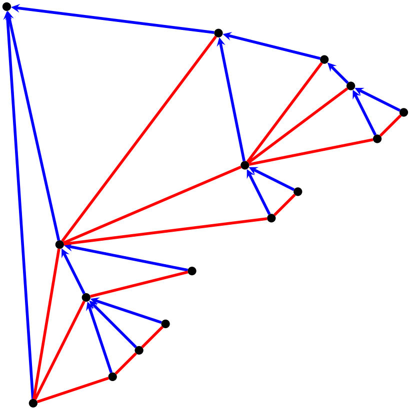
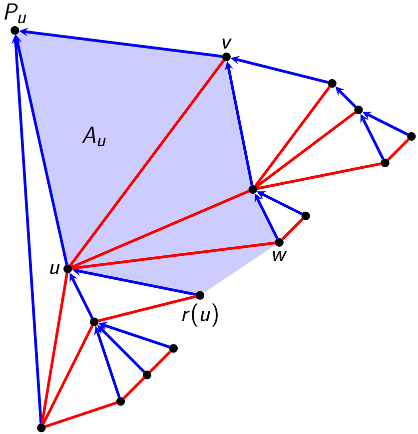
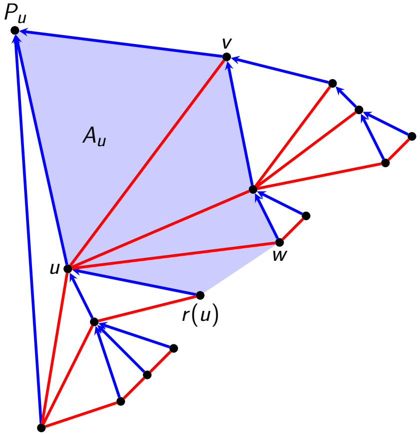




For each vertex u not a leaf of T_r : define an area A_u with only red edges leading to u (the areas A_i are disjoint).

Rook-drawing for outerplanar graphs

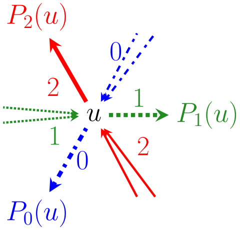


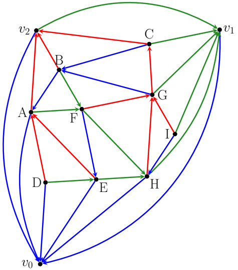
For each vertex u not a leaf of T_r : define an area A_u with only red edges leading to u (the areas A_i are disjoint).

The drawing is planar within A_u and blue and red edges can not cross.

Schnyder woods

A Schnyder wood is a partition of the edges of a triangulation in three trees T_0, T_1 and T_2 (directed toward the root) and with a particular configuration around each node :





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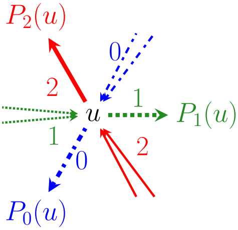
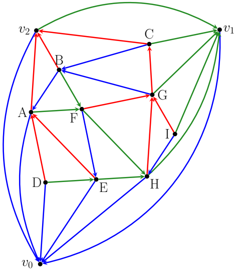




[Schnyder 1989]

Every plane triangulation admits at least one Schnyder wood, and it can be computed in linear time.

Polyline rook-drawing for planar graphs

We consider G a maximal plane graph (with exterior nodes v_0, v_1 and v_2).

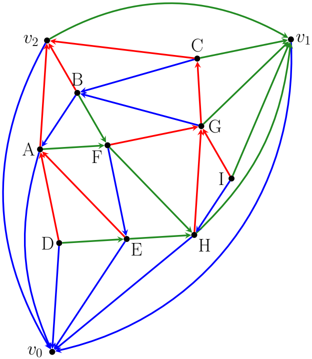
Main result

Every planar graph with n nodes admits a planar polyline rook-drawing, with at most $n-2$ bends (at most one per edge). Such a drawing can be computed in linear time.

Proof : based on an algorithm of [Bonichon, Mosbah, Le Saëc, 2002] optimizing the area of a polyline drawing.

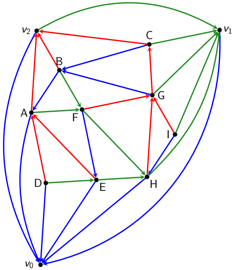
Planar polyline rook-drawing - Nodes

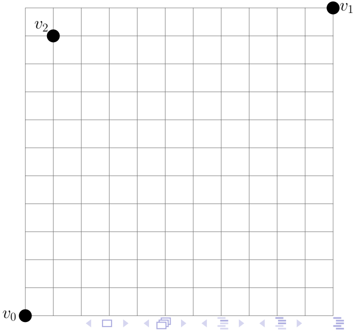
- (T_0, T_1, T_2) : Schnyder wood of G.



Planar polyline rook-drawing - Nodes

- (T_0, T_1, T_2) : Schnyder wood of G.
- $v_0 = (0, 0)$, $v_1 = (n-1, n-1)$, $v_2 = (1, n-2)$.





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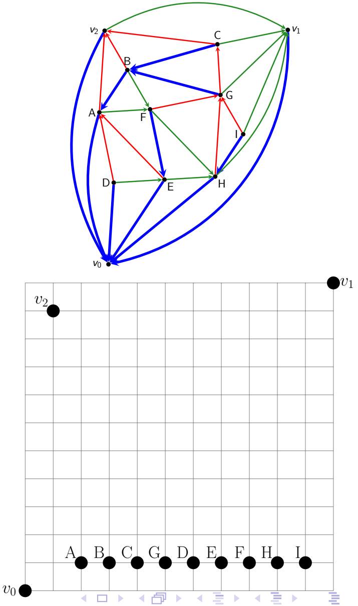

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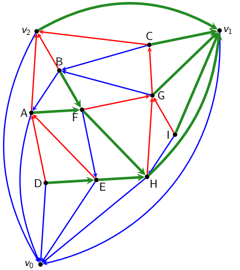

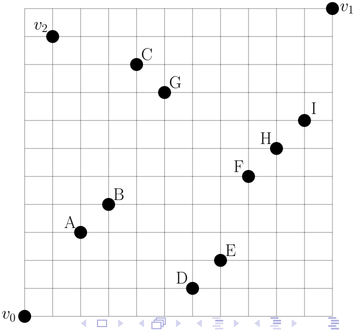

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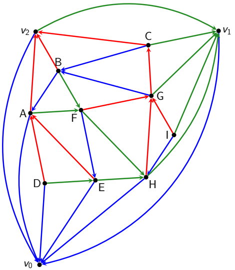
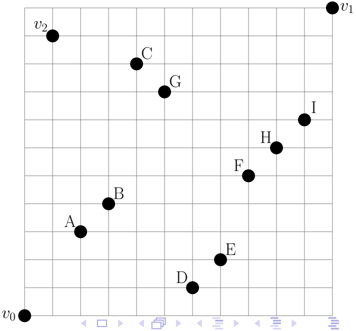
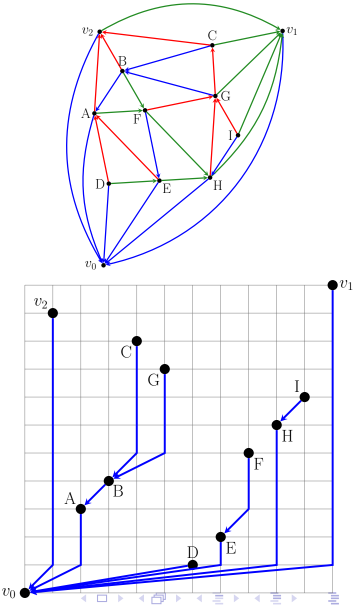




Planar polyline rook-drawing - Edges

- The edges $(u, P_0(u))$ are bent at $(x(u), y(P_0(u)) + 1)$



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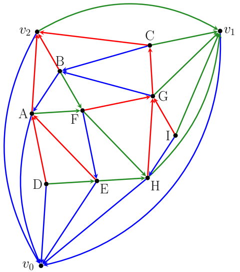

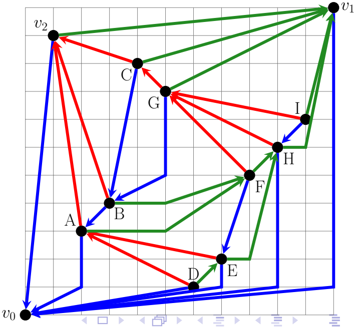

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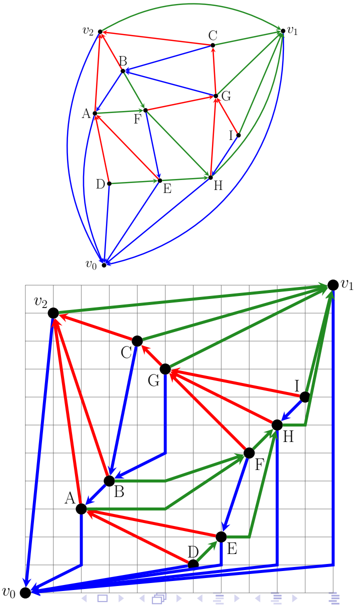
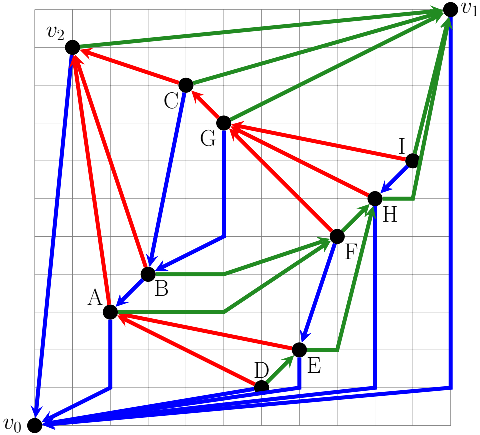
Around a node :
$\#bends = \#\{\text{blue children}\}$
$\rightarrow n - 2$ bends in the drawing

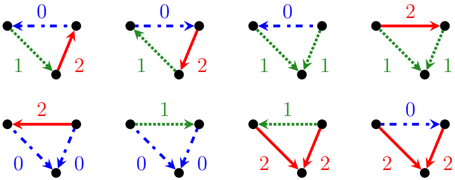


Proof of planarity



Schnyder wood properties

If u is a descendant of v in T_i, then u is not a parent of v in T_j, $j \neq i$.



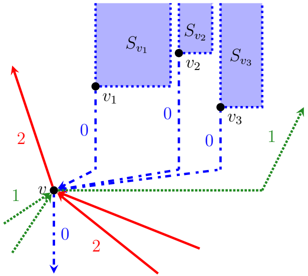
If u is parent of v in T_i, then u is before v in counterclockwise preordering of T_{i-1} and after v in counterclockwise preordering of T_{i+1}.

Edges direction

Edges direction

For each inner node u :

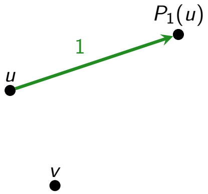
- $P_0(u)$ is left and below u.
- $P_1(u)$ is right and above u.
- $P_2(u)$ is left and above u.



- South-East : edges from children in T_2
- South : edge toward $P_0(v)$
- South-West : edges from children in T_1
- North-West : edge toward $P_2(v)$
- North-East : edges from children in T_0
- East : edge toward $P_1(v)$

Properties of green/red edges

Every node v with $x(u) < x(v) < x(P_1(u))$ has $y(v) < y(u)$ if v is not a descendant of u in T_0.



v : between u and $P_1(u)$.

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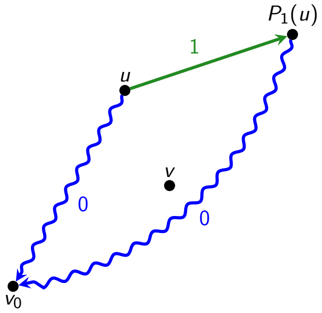

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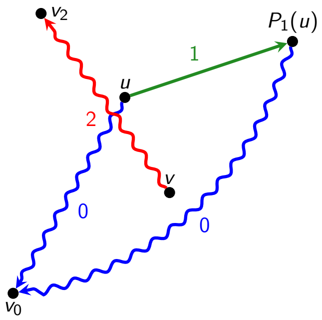

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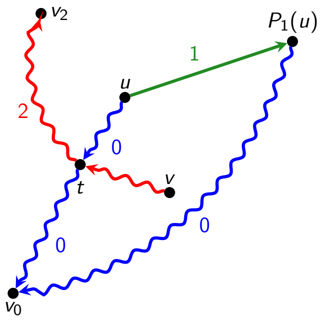

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Intersection t on path (v_0, u).

Paths $v \rightarrow t$ and $u \rightarrow t$.

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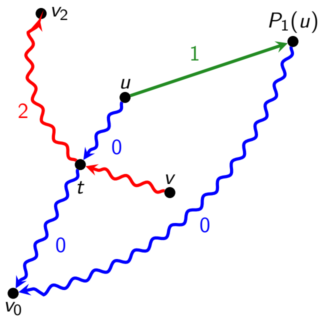

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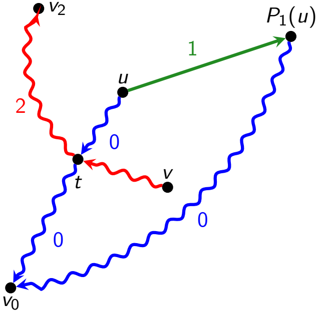

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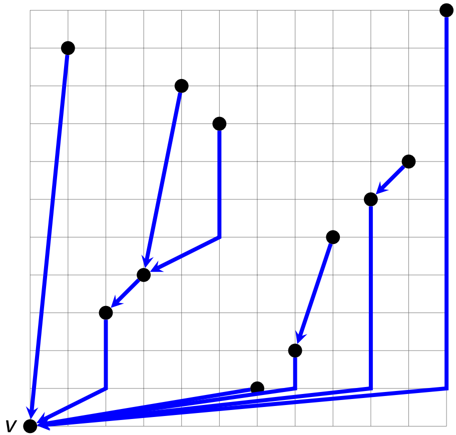
Every node v with $x(P_2(u)) < x(v) < x(u))$ has $y(v) < y(u)$ if v is not a descendant of $P_2(u)$ in T_0.

Non-crossing - blue

The edges of T_0 do not cross each other.



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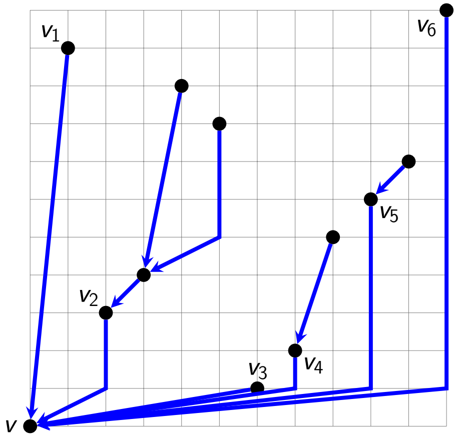

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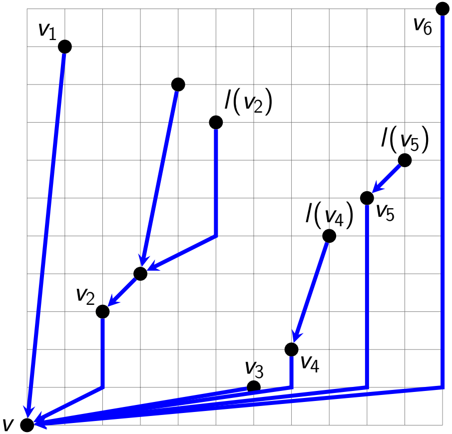

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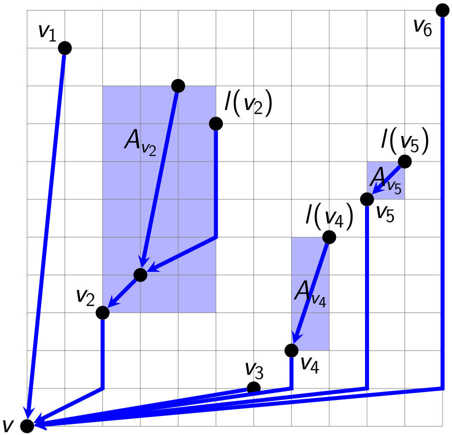

The subtrees of children "live" in different areas of width $(x(l(v_i)) - x(v_i))$.

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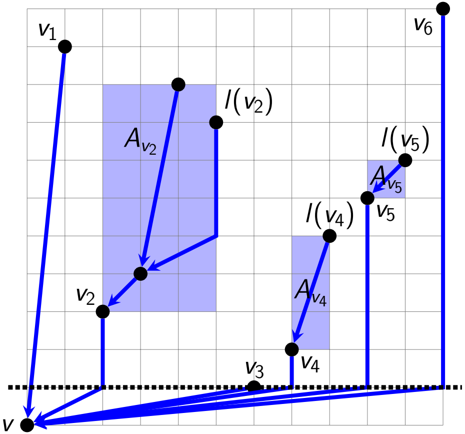

The subtrees of children "live" in different areas of width $(x(l(v_i)) - x(v_i))$.

The edges to the children can not cross each other.

Non-crossing - green

The edges of T_1 do not cross each other.

Subtrees live in different areas (by construction). The bends : y-decreasing (by construction) ; x-increasing :

Non-crossing - green

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u_i descendant of u_{i+1} in T_0

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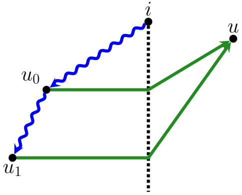

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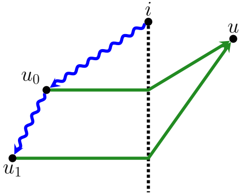

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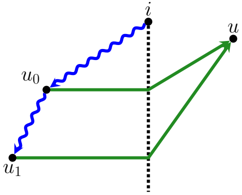

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u_i not descendant of u_{i+1} in T_0

$x(u_i) > x(u_{i+1})$

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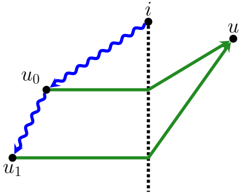

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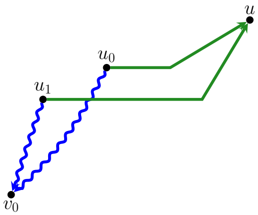

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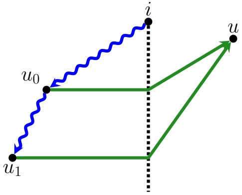

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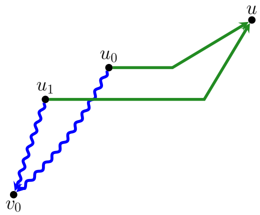

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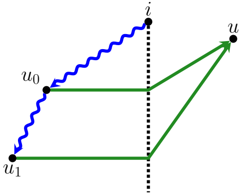

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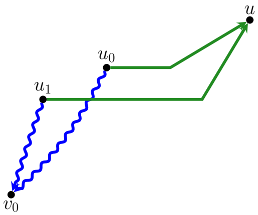

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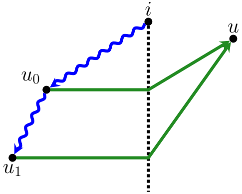

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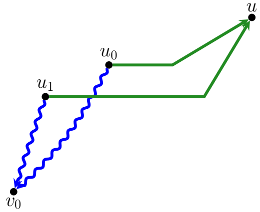

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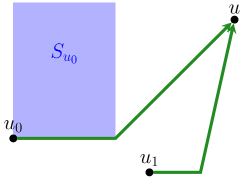

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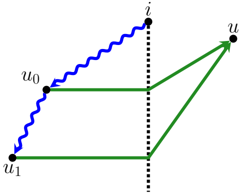

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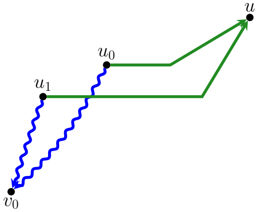
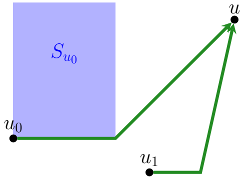
u_i not descendant of u_{i+1} in T_0

$x(u_i) > x(u_{i+1})$



u_i should be below the edge (u_{i+1}, u)

$x(u_i) < x(u_{i+1})$



Descendants of u_i are between $x(u_i)$ and $x(u_{i+1})$.

Conclusion

Open questions :

- Reduce the number of bends necessary to draw a given planar graph ?
- Caracterization of planar graphs for which a straight-lines rook-drawing is not possible / is possible
- What is the minimum grid size requested to draw a straight-lines rook-drawing ?

Thank you for your attention !